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Perturbative analysis of the interaction of a ϕ^4 kink with inhomogeneities

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Abstract. Dynamics of a kink in the ϕ^4 model with additional terms accounting for localized or spatially random inhomogeneities are analysed by means of the perturbation theory. First, conditions for the capture of the free kink by a local defect are found. Next, emission of radiation by the kink colliding with the local defect, and collision-induced generation of the kink's shape (internal) mode and the defect-sustained impurity mode are considered. Finally, the non-dissipative braking of the moving kink in a randomly inhomogeneous medium on account of the excitation of the shape mode is analysed.

1. Introduction

The subject of this work is dynamics of the kink in the perturbed ϕ^4 model, which has the following general form:

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = -\alpha\phi_t - f + \varepsilon_1(x)(\phi - \phi^3) - \varepsilon_2(x)\phi_{tt} \qquad (1)$$

where α is the dissipative constant, f is the external DC driving field, and the two last terms take account of a spatial inhomogeneity of the system. It is well known [1–3] that the ϕ^4 model describes the displacive phase transition in quasi-1D ferroelectrics. In this context, it arises as the continuum limit of a discrete system of interacting particles with the ϕ^4 on-site potential. This discrete model is also of some interest in itself, see e.g. [4]. In terms of these models, the perturbing terms on the right-hand side of equation (1) find their natural interpretation: f is the DC electric field applied to the system, α is a phenomenological lossy constant, and the coefficients $\varepsilon_1(x)$ and $\varepsilon_2(x)$ take account of the inhomogeneity of the, respectively, strength of the local on-site potential and mass of the particles. Physically, these inhomogeneities can be produced by imperfections of the underlying lattice or by impurities.

Finally, equation (1) with the inhomogeneity term corresponding to a localized defect (impurity)

$$\varepsilon_1(x) = \varepsilon\delta(x) \qquad \varepsilon_2 = 0 \qquad (2)$$

is an interesting model system for studying the kink–impurity interaction [5].

The kink solution to the unperturbed equation (1) is

$$\phi_k = \sigma \tanh((x-z)/\sqrt{2(1-V^2)}) \qquad (3)$$

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where $\sigma = \pm 1$ is the kink's polarity, V is its volocity and $z = Vt$ is the coordinate of its centre. The objective of the present work is to study, by means of the perturbation theory, the interaction of the kink with the local inhomogeneity corresponding to equation (2), or with that corresponding to

$$\varepsilon_1 = 0 \qquad \varepsilon_2 = \varepsilon\delta(x). \tag{4}$$

I will also consider motion of the kink through the random inhomogeneous medium described by equation (1) with the random functions $\varepsilon_{1,2}$ subject to the Gaussian correlations, i.e.

$$\langle \varepsilon_1(x) \rangle = 0 \qquad \langle \varepsilon_1(x)\varepsilon_1(x') \rangle = \varepsilon^2 \delta(x - x'). \tag{5}$$

It is well known [6] that the unperturbed ϕ^4 equation, linearized on the background of the kink solution, gives rise to the continuous spectrum of delocalized radiation modes, and to two localized modes constituting the discrete spectrum. One of the localized modes is the usual Goldstone one, related to the translational invariance of the ϕ^4 model, while the second mode describes small-amplitude internal shape oscillations of the kink. In the absence of the kink, the ϕ^4 equation linearized on the background of the localized defect (equations (2) or (4)) admits the so-called impurity mode (see e.g. [4]). Interaction of the kink with the localized or random inhomogeneities gives rise to excitation of the shape and radiation modes, as well as of the impurity mode.

In section 2, the interaction of the kink with the local inhomogeneity (2) is briefly considered in the adiabatic approximation, i.e. neglecting the excitation of the radiation, shape and impurity modes. Following the lines of the similar analysis for the sine-Gordon (SG) model [7, 8], the threshold (maximum) values of the drive parameter f in equation (1) admitting the capture of the kink by the inhomogeneity are found for positive and negative ε.

In section 3, the radiation losses of the kink scattered by the local defect (2) or (4) are investigated. The analysis is based on the Lagrangian technique employed earlier [9, 10] to analyse the excitation of the shape mode in the kink–antikink collision. The total emitted energy is calculated.

Section 4 is devoted to excitation of the shape and impurity modes in the kink–defect collision. This problem is of especial interest, as it is qualitatively different from the analysis developed earlier for the SG kink (see the review paper [11]). The SG kink has no shape mode, while its adiabatic interaction with local defects, as well as the collision-induced emission of radiation, have been studied in detail [11]. As for the shape mode, it exists in the SG model with a local defect, but the excitation of the shape mode has not previously been analysed (see also section 6). At the end of section 4, radiative and dissipative dampings of the excited shape and impurity modes are briefly considered.

In section 5, excitation of the shape mode of the kink moving in the randomly inhomogeneous medium described by equation (5) is studied. It is demonstrated that this is the shape mode which gives a dominant contribution to the non-dissipative braking of the kink in the inhomogeneous medium.

Finally, in the section 6 some new problems, which are related to those considered in the present work, are briefly discussed.

2. The adiabatic kink–impurity interaction

It is well known [3] that the kink (3) may be treated as a classical relativistic particle with the mass $m=\frac{4}{3}$ subject to the action of the driving and friction forces,

$$F_{\mathrm{dr}}=2\sigma f \tag{6}$$

$$F_{\mathrm{fr}}=-(2\sqrt{2}/3)\alpha V(1-V^2)^{-1/2}. \tag{7}$$

In the homogeneous system, the driven kink moves at the equilibrium velocity V_0 determined by the equation $F_{\mathrm{dr}}+F_{\mathrm{fr}}=0$ [3, 7]:

$$V_0(1-V_0^2)^{-1/2}=(2/\sqrt{3})\sigma(f/\alpha). \tag{8}$$

The local inhomogeneity (2) corresponds to the additional term

$$U\equiv\tfrac{1}{4}\varepsilon(\phi^2-1)^2|_{x=0} \tag{9}$$

in the full Hamiltonian of the ϕ^4 model. Inserting the kink's waveform (3) into equation (9), one finds the effective potential of the kink–impurity interaction [5]:

$$U(z)=\tfrac{1}{4}\varepsilon\,\mathrm{sech}^4(z/\sqrt{2}). \tag{10}$$

Thus, the local defect (2) repels or attracts the kink (irrespective of its polarity) if ε is, respectively, positive or negative. In the former case, it is straightforward to find the threshold (maximum) value f_{thr} of f at which the free kink coming across the defect will be captured by it. Following the lines of [7], one should equate the kinetic energy of the free 'non-relativistic' kink, $E_k=\frac{2}{3}V_0^2$, to the height $U_0\equiv\frac{1}{4}\varepsilon$ of the potential barrier (10). The eventual result is

$$f_{\mathrm{thr}}^2=(9/32)\varepsilon\alpha^2. \tag{11}$$

This result is valid under the condition that the force generated by the effective potential (10) must be much larger than the friction and driving forces at $z^2\lesssim 1$ [7]:

$$\varepsilon\gg\alpha^2. \tag{12}$$

In the opposite case ($\varepsilon<0$), when the local defect attracts the kink, one can find the value f_{thr} following the lines of [8]. Under the same condition (12), the threshold trajectory may be approximated by that corresponding to the motion of the particle in the potential (10) with zero velocity at infinity:

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\sqrt{-3\varepsilon/8}\,\mathrm{sech}^2(z/\sqrt{2}). \tag{13}$$

The energy dissipated by the kink on the trajectory (13) can be calculated as follows:

$$E_{\mathrm{diss}}\equiv-\int_{-\infty}^{+\infty}F_{\mathrm{fr}}\,\mathrm{d}z\approx(2\sqrt{2}/3)\alpha\int_{-\infty}^{+\infty}\dot{z}\,\mathrm{d}z=2\alpha\sqrt{-2\varepsilon/3}$$

where equation (7) has been used. Finally, equating the energy loss E_{diss} to the kinetic energy, one finds

$$f_{\mathrm{thr}}^2=\alpha^3\sqrt{-27\varepsilon/8}. \tag{14}$$

Note that, by virtue of the underlying condition (12), the threshold value (11) for the repulsive defect is much larger than that (14) for the attractive one. This fact is well known in the analysis of the similar problem in the SG model [8].

As concerns the local defect described by equation (4), the corresponding energy of the kink–defect interaction is (cf equation (10))

$$E_{\mathrm{int}}=\tfrac{1}{2}\varepsilon V^2(1-V^2)^{-1}\,\mathrm{sech}^2(z/\sqrt{2}). \tag{15}$$

Since the interaction energy (15) vanishes when $V=0$, this defect cannot capture the kink.

3. Emission of radiation in the kink–impurity collision

Using the results of [6], in the linear approximation one can represent the radiation wavefield on the background of the kink in the following form:

$$\begin{aligned}\phi(x',t')-\phi_k(x')&=\int_{-\infty}^{+\infty}dk'[8\pi(2+k'^2)(1+2k'^2)]^{-1/2}\\&\times\{[3\tanh^2(x'/\sqrt{2})-3\sqrt{2}\,ik'\tanh(x'/\sqrt{2})-(1+2k'^2)]\\&\times\exp[i(k'x'-\sqrt{2+k'^2}\,t')]b(k')+\mathrm{cc}\}\end{aligned}\qquad(16)$$

where ϕ_k is the kink waveform, $b(k')$ are the complex *spectral amplitudes* of the emitted radiation, and the coordinate x', the time t' and the wavenumber k' pertain to the reference frame moving together with the kink. If the ϕ^4 equation contains a small perturbing term $P[\phi]$ which can be deduced from an additional term in the full Lagrangian (e.g. $P[\phi]=\varepsilon_1(x)(\phi-\phi^3)-\varepsilon_2(x)\phi_{tt}$ in equation (1)), the radiation wavefield (16) gives rise to the following terms in the full Lagrangian:

$$\begin{aligned}L_{\mathrm{rad}}=\frac{i}{2}\int_{-\infty}^{+\infty}dk'\sqrt{2+k'^2}[b^*(k')\dot{b}(k')-\mathrm{cc}]+\frac{1}{4}\int_{-\infty}^{+\infty}dk'\\\times[\pi(2+k'^2)(1+2k'^2)]^{-1/2}\int_{-\infty}^{+\infty}dx'\,P[\phi_k(x')]\\\times\{[3\tanh^2(x'/\sqrt{2})+3\sqrt{2}\,ik'\tanh(x'/\sqrt{2})-(1+2k'^2)]\\\times\exp[-i(k'x'-\sqrt{2+k'^2}t')]b^*(k')+\mathrm{cc}\}\end{aligned}\qquad(17)$$

where the overdot stands for the time derivative. The first term in equation (17), quadratic in the spectral amplitudes, comes from the unperturbed part of the ϕ^4 Lagrangian (evidently, the term of the unperturbed Lagrangian linear in the spectral amplitudes makes no contribution to the action of the system). Varying the Lagrangian (17) in $b(k')$ yields the following linearized equations of motion for the spectral amplitudes:

$$\begin{aligned}\dot{b}(k')=\frac{i}{2}(2+k'^2)^{-1}[\pi(1+2k'^2)]^{-1/2}\exp(i\sqrt{2+k'^2}\,t')\\\times\int_{-\infty}^{+\infty}dx'\,P[\phi_k(x')][3\tanh^2(x'/\sqrt{2})+3\sqrt{2}\,ik'\tanh(x'/\sqrt{2})\\-(1+2k'^2)]\exp(-ik'x').\end{aligned}\qquad(18)$$

Assuming no radiation prior to the collision, one can define the *final values* [11] of the collision-generated amplitudes as follows:

$$[b(k')]_{\mathrm{fin}}=\int_{-\infty}^{+\infty}dt'\,\dot{b}(k').\qquad(19)$$

The basic physical characteristic of the emitted radiation is the spectral density $\mathscr{E}(k')$ of the radiation energy. Considering the linearized ϕ^4 equation, one can readily find that, in the lowest approximation,

$$\mathscr{E}(k)')=\sqrt{2}(2+k'^2)|b_{\mathrm{fin}}(k')|^2.\qquad(20)$$

Finally, one can define the total energy of the emitted radiation [11],

$$E_{\text{rad}} = \int_{-\infty}^{+\infty} \mathscr{E}(k')\,\mathrm{d}k'. \tag{21}$$

In the reference frame moving together with the kink, the perturbing term corresponding to equation (2) takes the form

$$P = \varepsilon\sqrt{1-V^2}\,\delta(x+Vt)(\phi-\phi^3). \tag{22}$$

Inserting equation (22) into equation (18) and performing the subsequent calculations according to equations (19) and (20), one finds that, in the most interesting 'ultrarelativistic' limiting case, $1-V^2 \ll 1$, the spectral density of the emitted energy is

$$\mathscr{E}(k') \approx (\pi/16)\varepsilon^2(1-V^2)^5(k')^4/\sinh^2(\pi(1-V^2)k'/2\sqrt{2}). \tag{23}$$

In the laboratory reference frame, the wavenumber k and the frequency $\omega \equiv \sqrt{1+k^2}$ are related to those k' and $\omega' \equiv \sqrt{1+k'^2}$ by the Lorentz transformation:

$$k' = (k - V\omega)/\sqrt{1-V^2} \equiv (k - V\sqrt{1+k^2})/\sqrt{1-V^2}. \tag{24}$$

The Lorentz transformation of the energy and momentum (the momentum spectral density is $\mathscr{P}(k') = (k'/\omega')\mathscr{E}(k')$) demonstrates that the energy spectral density is the Lorentz scalar (i.e. it is invariant with respect to the Lorentz transformation). Thus, substituting equation (24) for k' in equation (23) yields the expression for the energy spectral density in the laboratory reference frame. Finally, integrating it over $\mathrm{d}k$ (cf equation (21)) gives the total emitted energy in the laboratory reference frame,

$$E_{\text{rad}} = (2/15)\varepsilon^2\sqrt{1-V^2}. \tag{25}$$

Note that the analysis of the interaction of the SG kink with the local defect described by the term $\varepsilon\delta(x)\sin\phi$ in the perturbed SG equation demonstrates the same dependence, $E_{\text{rad}} \sim \sqrt{1-V^2}$, in the limit $1-V^2 \to 0$ [11].

In the case of the local defect corresponding to equation (4), the difference from the previous expressions for the energy spectral density and the total emitted energy amounts to the additional factor $V^4(1-V^2)^{-2}$. In particular, in the limit $1-V^2 \to 0$ the total emitted energy diverges $\sim(1-V^2)^{-3/2}$, cf equation (25).

4. Excitation of the impurity and shape modes

4.1. *The impurity mode*

It is straightforward to find that the local inhomogeneity (2) supports the impurity mode which, in the linear approximation, has the form

$$\psi \equiv \phi \pm 1 = B\exp(\mathrm{i}\omega t + \varepsilon|x|) + \text{cc} \tag{26}$$

where B is the small amplitude, and the frequency

$$\omega = \sqrt{2-\varepsilon^2} \approx \sqrt{2} - \varepsilon^2/2\sqrt{2}. \tag{27}$$

The impurity mode (26) exists only at $\varepsilon < 0$, when the impurity is attractive (see equation (10)). Note that only in this case is the mode (26) localized ($\exp(\varepsilon|x|) \to 0$ at $|x| \to \infty$).

Assuming that the coefficient ε in front of the local inhomogeneity is small, one expects the collision of the kink with the inhomogeneity to excite the impurity mode with a sufficiently small ampltitude. This process is governed by the perturbed ϕ^4 equation, the left-hand side of which is linearized in ψ:

$$\psi_{tt}-\psi_{xx}-\psi+3\phi_k^2\psi=\varepsilon\delta(x)(\phi_k-\phi_k^3). \tag{28}$$

The right-hand side of equation (28), where only the contribution from the unperturbed kink is retained, plays the role of the drive (source) exciting the impurity mode. Next, to derive an evolution equation for the amplitude B of the impurity mode, one should mutiply equation (28) by $\exp(\varepsilon|x|)$ (see equation (26)) and integrate it over dx from $-\infty$ to $+\infty$. The important circumstance is that, since $|\varepsilon|$ is small, the eigenmode (26) is weakly localized: its characteristic size $\sim|\varepsilon|^{-1}$ is much larger than the proper size of the kink, which is of order one in the notation adopted. Therefore, when integrating over dx, one may in the crudest approximation substitute the coefficients $3\phi_k^2$ in front of the last term on the left-hand side of equation (28) by the constant coefficient equal to 3 (evidently, this substitution is irrelevant for the right-hand side of equation (28)). After this, one can readily derive the following evolution equation for the impurity-mode amplitude B:

$$\dot{B}=-\mathrm{i}(2\sqrt{2})^{-1}\varepsilon^2\,\mathrm{e}^{-\mathrm{i}\omega t}\sinh(Vt/\sqrt{2(1-V^2)})\,\mathrm{sech}^3(Vt/\sqrt{2(1-V^2)}). \tag{29}$$

Finally, integrating equation (29), one finds the final value (cf equation (19)) of the amplitude:

$$B_{\mathrm{fin}}\equiv\int_{-\infty}^{+\infty}\mathrm{d}t\,\dot{B}=-\pi\varepsilon^2(1-V^2)^{3/2}V^{-3}/\sinh(\pi\sqrt{1-V^2}/V). \tag{30}$$

The energy E_{im} of the impurity mode (26) can be readily found in the lowest approximation:

$$E_{\mathrm{im}}\approx\int_{-\infty}^{+\infty}\mathrm{d}x(\tfrac{1}{2}\Psi_t^2+\Psi^2)\approx4B_{\mathrm{fin}}^2/|\varepsilon|. \tag{31}$$

In the case of the local inhomogeneity (4), the impurity mode exists provided ε is positive, and it has the form (cf equation (26))

$$\psi\equiv\phi\pm1=B\,\mathrm{e}^{\mathrm{i}\omega t-\varepsilon|x|}+\mathrm{cc} \tag{32}$$

where the frequency ω is the same as in equation (27). Note that, according to equation (15), positive ε corresponds to the repulsive defect. Thus, in case (4) the impurity mode is supported by the repulsive impurity, in contrast to case (2), when the supporting inhomogeneity must be attractive. The calculation of the final value of the amplitude B of the impurity mode (32) is quite similar to that performed above for case (2), and the eventual expression for B_{fin}^2 differs by the additional multiplier $V^4(1-V^2)^{-2}$. In particular, equation (31) tells, us with regard to the additional multiplier, that in the limit $1-V^2\to0$ the energy of the excited impurity mode takes the finite value $4\varepsilon^3$. Note that, in the general case, the energy expended on the generation of the impurity mode is $\sim|\varepsilon|^3$, while the radiation energy losses are $\sim\varepsilon^2$ according to equation (25). Thus, the impurity-mode losses remain negligible in comparison with the radiation ones as long as the parameter ε remains small.

4.2. The shape mode

If the shape mode is excited, the disturbed kink can be represented in the form [6]

$$\phi(x', t') - \phi_k(x') = (\sqrt{3}/2)\,\mathrm{sech}(x'/\sqrt{2})\tanh(x'/\sqrt{2})[e^{-i\sqrt{3/2}t'}\, b_0(t') + \mathrm{cc}] \tag{33}$$

(cf equation (16)). The parts of the full Lagrangian of the perturbed ϕ^4 model, linear and quadratic in the shape-mode amplitude $b_0(t')$ (cf equation (17)), are

$$L_{\mathrm{shape}} = -\frac{i}{4}\sqrt{3/2}(b_0\dot{b}_0^* - \mathrm{cc}) + \sqrt{3/2}(\varepsilon/2)\int_{-\infty}^{+\infty} dx'\, P[\phi_k(x')] \times \mathrm{sech}(x'/\sqrt{2})\tanh(x'/\sqrt{2})(e^{i\sqrt{3/2}t'}\, b_0^* + \mathrm{cc}). \tag{34}$$

Varying the expression (34) in b_0 yields the following evolution equation (cf equation (18)):

$$\dot{b}_0 = i\sqrt{2}\,\varepsilon \int_{-\infty}^{+\infty} dx'\, P[\phi_k(x')]\,\mathrm{sech}(x'/\sqrt{2})\tanh(x'/\sqrt{2})\, e^{i\sqrt{3/2}t'}. \tag{35}$$

The final value of the shape-mode amplitude can be defined exactly as in equation (19):

$$(b_0)_{\mathrm{fin}} = \int_{-\infty}^{+\infty} dt'\, \dot{b}_0. \tag{36}$$

Note that this way of investigating the excitation of the shape mode of the kink colliding with the local defect is essentially similar to the approach to the same problem for the kink–antikink collision developed in [9] (see also [10]) and based on the collective-coordinate technique.

It is straightforward to express the energy of the shape oscillations in terms of the amplitude:

$$E_{\mathrm{shape}} = (9/10)|(b_0)_{\mathrm{fin}}|^2. \tag{37}$$

For the local defect corresponding to equation (2), the calculations based on equations (35) and (36) yield

$$|(b_0)_{\mathrm{fin}}|^2 = (\varepsilon/4)^2 V^{-10}(3+V^2)^2(1-V^2)^3\,\mathrm{sech}^2(\pi\sqrt{3}/2V). \tag{38}$$

For the defect corresponding to equation (4), expression (38) should be multiplied by $V^4(1-V^2)^{-2}$. Comparing equation (38) with equation (25), one concludes that, in the limiting case $1 - V^2 \to 0$, the energy expended on the excitation of the shape mode becomes negligible in comparison with the radiation losses.

4.3. The damping of the shape and impurity modes

The shape oscillations are subject to radiative damping. In fact, this effect has already been considered in [9]. The radiative damping is accounted for by the term in the equation quadratic in the shape-oscillation amplitude b_0. Indeed, the quadratic term gives rise to the shape oscillations at the double frequency $2\omega_{\mathrm{shape}} = 2(\sqrt{3/2}) \equiv \sqrt{6}$, which couples with the radiation wavenumbers

$$k = \pm\sqrt{(2\omega_{\mathrm{shape}})^2 - 2} \equiv \pm 2.$$

The calculations based on the general evolution equation (18) and on expression (20) for the energy spectral density yield the corresponding energy emission rate

$$W = [2\times 3^{7/2}\pi/\sinh^2(\sqrt{2}\pi)]b_0^4. \tag{39}$$

Expression (39) is quartic in the amplitude b_0 because the energy emission rate is quadratic in the emission amplitude, and, as was said above, the latter is quadratic in b_0.

The energy-balance equation

$$\frac{d}{dt} E_{\text{shape}} = -W \tag{40}$$

determines the rate of the radiative damping of the shape oscillations. Inserting equations (37) and (39) into equation (40), one sees that the damping law takes the form

$$b_0^2 \sim t^{-1} \tag{41}$$

i.e. the damping is non-exponential. For comparison, the dissipative term in equation (1) gives rise to the exponential damping law

$$b_0^2 \sim \exp(-(10\sqrt{2}/9)\alpha t). \tag{42}$$

To conclude this section, let us note that the impurity mode is also subject to radiative and dissipative damping. The corresponding energy-balance equations take the following form:

$$\frac{d}{dt}|B|^2 = -(2\sqrt{3})^{-1}|\varepsilon|^3|B|^4$$

$$\frac{d}{dt}|B|^2 = -\alpha|B|^2$$

for the cases of radiative and dissipative dampings, respectively (recall that B is the amplitude of the impurity mode (26)).

5. Excitation of the shape mode in the randomly inhomogeneous medium

Let us consider free motion of the kink in the medium described by equation (5). The random inhomogeneity gives rise to excitation of the shape mode and, simultaneously, to emission of radiation. The most interesting issue is braking of the kink due to the losses of energy expended on generation of the shape oscillations and radiation waves. At the final stage of the braking, the shape-mode losses play a dominant role. The rate W of the transfer of energy into the shape mode can be obtained from equation (37):

$$W \equiv \frac{d}{dt} E_{\text{shape}} = \tfrac{9}{5}\,\text{Re}(\dot{b}_0 b_0^*). \tag{43}$$

The evolution equation (35) in the reference frame moving together with the kink is

$$\dot{b}_0(t') = i\sqrt{2}\int_{-\infty}^{+\infty} dx'\,\varepsilon_1((x'+Vt')/\sqrt{1-V^2})\tanh^2(x'/\sqrt{2})\,\text{sech}^2(x'/\sqrt{2})\,e^{i\sqrt{3/2}t'}. \tag{44}$$

The formal integral of equation (44) is

$$b_0(t') = i\sqrt{2}\int_{-\infty}^{t'} dt'' \int_{-\infty}^{+\infty} dx''\,\varepsilon_1((x''+Vt'')/\sqrt{1-V^2})\tanh^2(x''/\sqrt{2}) \times \text{sech}^3(x''/\sqrt{2})\,e^{i\sqrt{3/2}t''}. \tag{45}$$

Inserting equations (44) and (45) into equation (43) and averaging the result according to equation (5), one can obtain the following expression in the limiting case $V \ll 1$, corresponding to the final stage of the braking:

$$W \approx \tfrac{1}{2}\varepsilon^2 V^{-8}\, \mathrm{e}^{-\pi\sqrt{3}/V}. \tag{46}$$

The energy-balance equation for the kink is (recall that, in the notation adopted, the mass of the kink is $m = 4/3$):

$$\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t} V^2 = -W. \tag{47}$$

Inserting equation (46) into equation (47) yields the asymptotic law of motion of the kink:

$$V(t) \approx \pi\sqrt{3}/\ln(\varepsilon^2 t). \tag{48}$$

Formally, the integral $z(t) \equiv \int V(t)\mathrm{d}t$ corresponding to equation (48) diverges at $t \to \infty$, so that the kink does not stop at any finite value of z. However, the kink will actually be captured by a local potential well, induced by the random inhomogeneity, when $V^2(t)$ diminishes to a value $\sim\varepsilon$; cf the similar situation in the analysis of the radiative braking of the SG kink moving through a lattice of local inhomogeneities [11].

As for the radiation losses, in the case $V \ll 1$ the energy emission rate contains the factor $\exp(-2\pi/V)$, hence it may be neglected in comparison with expression (46). Note that the radiation losses of the SG kink in the randomly inhomogeneous medium were first studied in [12].

6. Conclusion

The problems considered in the present work can be generalized in different directions. For instance, one can consider the SG model with the local defect,

$$\phi_{tt} - \phi_{xx} + \sinh\phi = -\varepsilon\delta(x)\sin\phi. \tag{49}$$

Model (49) has a number of physical applications. For example, it describes a long Josephson junction with an installed microresistor ($\varepsilon < 0$) or microshort ($\varepsilon > 0$) [7]. Interaction of the SG kink with the local defect has been studied in detail [11], except for the generation of the impurity mode. In the linear approximation, the impurity mode is (cf equation (26))

$$\phi = B\, \mathrm{e}^{\mathrm{i}\omega t + (\varepsilon/2)|x|} + \mathrm{CC} \tag{50}$$

where $\omega = \sqrt{1-(\varepsilon/2)^2} \approx 1 - \varepsilon^2/8$ (cf equation (27)). The impurity mode (50) exists only in the case $\varepsilon < 0$, ie.. when the local defect attracts the kink [7]. Note that exactly the same situation has been encountered above in the ϕ^4 model based on equations (1) and (2). The calculation of the collision-induced amplitude of the impurity mode is quite similar to that performed in section 4.1. The final result is (cf equation (30))

$$B_{\mathrm{fin}} = -(\mathrm{i}\pi/2)\varepsilon^2(1-V^2)V^{-2}\,\mathrm{sech}(\pi\sqrt{1-V^2}/2V)$$

and the energy of the impurity mode (50) is (cf equation (31))

$$E_{\mathrm{im}} \approx \int_{-\infty}^{+\infty} \mathrm{d}x(\tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi^2) = 4B_{\mathrm{fin}}^2/\varepsilon.$$

Apart from the kink, the unperturbed SG equation has the soliton solution in the form of the so-called breather:

$$\phi_{br}=4\tan^{-1}[\tan\mu\cos(\cos\mu\,(t-Vx)/\sqrt{1-V^2})\times\operatorname{sech}(\sin\mu\,(x-Vt)/\sqrt{1-V^2})] \quad (51)$$

where μ takes the values $0<\mu<\pi/2$. The collision of the breather with the local defect gives rise to emission of radiation, the total emitted energy being

$$E_{rad}\sim\varepsilon^2/V \quad (52)$$

unless the breather's internal frequency $\omega_{br}\equiv\cos\mu$ is close to the resonant value $\omega_{br}=\frac{1}{3}$. In the latter case, the triple frequency $3\omega_{br}$ gets close to the edge $\omega_0=1$ of the radiation spectrum $\omega(k)\equiv\sqrt{1+k^2}$. It is known that in this case the radiation effects are essentially stronger than far from the resonance [11]. The final estimate for the emitted energy in this case is (cf equation (52)) $E_{rad}\sim\varepsilon^2/V^{3/2}$, where it is implied that the velocity V is sufficiently small; the emitted energy is concentrated in the spectral region $k^2\lesssim V$.

An interesting problem is to consider the interaction of the kink with the impurity and shape modes in the case when the kink's kinetic energy is of the same order of magnitude as the energies of the excited modes. In this case, the interaction can give rise to rich dynamics. In particular, the regions of the values of the initial velocity of the kink, corresponding to the capture, transmission and reflection of the kink colliding with the local defect, can alternate in a complicated way [5, 13, 14].

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